

# Cosmological consequences of generalised RS II braneworlds

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We discuss certain features of cosmology in a generalised RS II braneworld scenario. In this scenario, the bulk is given by a Schwarzschild-anti de Sitter or a Vaidya-anti de Sitter black hole in which an FRW brane is consistently embedded, resulting in modifications of the 4-dimensional Friedmann equations. We analyse how the scenario can be visualised and discuss the significance of each term in these modified equations both for early time and for late time cosmology. We further analyse the perturbation equations, based on Newtonian as well as relativistic perturbations and show that the scenario has the potentiality to explain structure formation by the “Weyl fluid” arising from embedding geometry. The results thus obtained are confronted with observations as well.

## INTRODUCTION

The Randall-Sundrum braneworld model [1] provides us with a unique mechanism of solving the hierarchy problem and obtaining an effective 4-dimensional Newton’s law of gravity valid at large as well as small length scales, by the method of embedding our 4D universe in a higher dimensional anti-de Sitter ( $AdS_5$ ) space. Of late, a more general, and geometrically rich, picture of obtaining an effective 4D theory of gravity, by projecting the bulk field equations onto the brane, has been formulated by Shiromizu et al [2]. In this scenario, the bulk space-time is not necessarily  $AdS_5$ , rather a generalised version of it. The so-called *effective Einstein equation on the brane*, thus obtained, not only gives important insight to the bulk-brane interplay but also raises the possibility of explaining the gravitational phenomena of the 4D universe from a broader perspective. Since the formalism was brought forth, it has been applied to explain several gravitational aspects, leading to interesting results which justify further investigation on the prospects of braneworlds. A thorough discussion of these issues is available in the review of S. Kar [3] in this volume.

The cosmological aspects of this theory were developed at a rather later stage, due to the complicated equations arising from the extra terms involved in the theory, which makes the theory difficult to probe at a first go. The basic aim of this article is to provide a brief review of what has been done in the cosmological context of braneworlds so far and show that, in spite of great complications involved, the cosmological consequences of this scenario results in interesting physics which is worth investigating further.

## FRW BRANE, SCH-ADS BULK

In the braneworld scenario, the bulk may be either empty with only a bulk cosmological constant, or it may consist of non-standard model fields minimally or non-minimally coupled to gravity or to brane matter. For empty bulk, the bulk metric in which an FRW brane can be consistently embedded, is given by a 5-dimensional Schwarzschild-Anti de Sitter (Sch- $AdS_5$ ) black hole [4, 5]

$$dS_5^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_3^2 \quad (1)$$

where  $\Sigma_3$  is the 3-space and the function  $f(r)$  is given by

$$f(r) = k - \frac{\Lambda_5}{6}r^2 - \frac{m}{r^2} \quad (2)$$

where  $k(= 0, \pm 1)$  is the spatial curvature. Here  $m$  is a constant which is identical to the 5D analogue of the Schwarzschild mass of the bulk black hole.

An important point to note here is that the bulk metric presented here is asymptotically anti-de Sitter. When the mass of the 5D black hole vanishes, this metric can be recast in the warped RS II form by coordinate transformation. Therefore, the scenario is a generalisation of the RS II braneworlds, which reflects cosmology on the brane. In this sense, this setup may be called *generalised RS II braneworlds*.

The Sch- $AdS_5$  black hole provides a novel way of visualising cosmological phenomena on the 4D universe. In this scenario, the brane is moving in the bulk, with its radial trajectory given by  $r(\tau)$ , where  $\tau$  is the proper time on the brane. With  $u^\mu = (\dot{t}, \dot{r})$  the velocity vector on the brane, the normalisation condition is given by

$$g_{\mu\nu}u^\mu u^\nu = -f\dot{t}^2 + \frac{\dot{r}^2}{f} = -1 \quad (3)$$

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and the unit normal vector  $n^\mu$ , satisfying the orthonormality conditions, is given by

$$n^\mu = (\dot{r}, -\frac{\sqrt{f + \dot{r}^2}}{f}) \quad (4)$$

With this, we can get the induced metric on the brane

$$ds^2 = -d\tau^2 + r^2(\tau)d\Omega_3^2 \quad (5)$$

which is, clearly, an FRW metric on the brane, with the scale factor  $a(\tau)$  being identified with the radial trajectory  $r(\tau)$ . So, the interpretation here is as follows: What we call an expanding universe is visualised by a brane-based observer only. For a bulk-based observer located somewhere outside the bulk black hole horizon, this expansion is exactly identical to the movement of the brane along the radial trajectory of the black hole.

Given this situation, the important question is: How do the Friedmann equations on the brane look like? The review of S. Kar in this volume [3] discusses in details that the 4D Einstein equation on the brane is a generalisation of the standard Einstein equation. The so-called *Effective Einstein Equation* is given by [2]

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa_4^2 T_{\mu\nu} + \kappa_5^4 \mathcal{S}_{\mu\nu} - \mathcal{E}_{\mu\nu} \quad (6)$$

where the 4D cosmological constant and coupling constant are related to their 5D counterparts by

$$\Lambda = \frac{\kappa_5^2}{2} \left( \Lambda_5 + \frac{\kappa_5^4 \lambda_b^2}{6} \right), \quad \kappa_4^2 = \frac{\kappa_5^2 \lambda_b}{6} \quad (7)$$

$\mathcal{S}_{\mu\nu}$  is the quadratic contribution from brane energy-momentum tensor

$$\mathcal{S}_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T_{\mu\alpha} T_\nu^\alpha + \frac{1}{8} T_{\alpha\beta} T^{\alpha\beta} g_{\mu\nu} - \frac{1}{24} T^2 g_{\mu\nu} \quad (8)$$

and  $\mathcal{E}_{\mu\nu}$  is the bulk Weyl tensor, projected onto the brane, which is given by

$$\mathcal{E}_{\mu\nu} = C_{ABCD} n^C n^D g_\mu^A g_\nu^B \quad (9)$$

Thus, along with the usual contributions from the cosmological constant and matter on the 4D universe, the EEE contains two additional terms :

- $\mathcal{S}_{\mu\nu}$  : the local correction from brane matter
- $\mathcal{E}_{\mu\nu}$  : the nonlocal correction from bulk geometry.

For a perfect fluid on the brane, using the symmetry properties, these braneworld corrections can be added up to the usual brane energy-momentum tensor. The effective energy density, pressure, momentum density and anisotropic stress thus obtained are, respectively, given by [4]:

$$\rho^{\text{eff}} = \rho + \frac{\rho^2}{2\lambda_b} + \rho^* \quad (10)$$

$$p^{\text{eff}} = p + \frac{\rho}{2\lambda_b}(\rho + 2p) + \frac{\rho^*}{3} \quad (11)$$

$$q_\mu^{\text{eff}} = q_\mu^* \quad (12)$$

$$\pi_{\mu\nu}^{\text{eff}} = \pi_{\mu\nu}^* \quad (13)$$

Further, the brane matter conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \quad (14)$$

leads to the conservation equation for the Weyl term (via 4D Bianchi identity on the brane  $\nabla^\mu G_{\mu\nu} = 0$ )

$$\dot{\rho}^* + 4\frac{\dot{a}}{a}\rho^* = 0 \quad (15)$$

Hence the Weyl density  $\rho^*$  behaves as

$$\rho^* = \frac{C}{a^4} \quad (16)$$

where  $C$  is a constant which is basically the rescaled bulk black hole mass.

Further, for a bulk compatible to FRW geometry on the brane,  $q_\mu^* = 0 = \pi_{\mu\nu}^*$ . With the use of the effective Einstein equation (6) and the effective quantities obtained in equations (10) and (11) for a perfect fluid on the brane, the Friedmann equation and the covariant Raychaudhuri equation turn out to be

$$H^2 = \frac{\kappa_4^2}{3}\rho^{\text{eff}} + \frac{\Lambda}{3} - \frac{k}{a^2} \quad (17)$$

$$\dot{H} = -\frac{\kappa_4^2}{2}(\rho^{\text{eff}} + p^{\text{eff}}) + \frac{k}{a^2} \quad (18)$$

Obviously, these equations are quite different from the standard 4D equations due to the presence of the quadratic terms and the Weyl terms.

Let us now explore the roles of these additional terms on 4D cosmology. The quadratic term  $\mathcal{S}_{\mu\nu}$  plays a significant role in the early universe ( $\rho^2 \gg \lambda_b$ ). For example, this leads to a faster Hubble expansion at high energies and a more strongly damped evolution of the inflaton field [6]. Thus the brane universe inflates at a much faster rate than what is expected from standard cosmology. Further, with this term, the modified equations make it possible to explain the inflationary scenario which is driven not by an inflaton field on the brane (*i.e.*, not by any 4D field as required in standard cosmology) but by a dilaton field in the bulk [7]. However, the quadratic term is negligibly small at late times as  $\rho^2 \ll \lambda_b > (100\text{GeV})^4$ . Hence, its role is relevant so far as only the early universe is concerned.

On the other hand, the role of  $\mathcal{E}_{\mu\nu}$  is to supply an additional perfect fluidlike effect to the actual on-brane perfect fluid (since, from the previous arguments,  $q_\mu^* = 0 = \pi_{\mu\nu}^*$ ). The so-called *Weyl fluid* arises as the tidal effect of the 5D black hole in the bulk and its density contribution  $\rho^*$  is related to the mass of the bulk black hole. Hence, in order to have a realistic contribution from the Weyl term on the brane, we need the black hole mass to be positive, so that  $\rho^* > 0$ . Since for a vacuum bulk, Eq (16) reveals that the Weyl fluid is strictly radiation-like,

it does not play any significant role in late time cosmologies. It can, at best, slightly modify the standard perturbative analysis. Its role has been extensively studied for metric-based perturbations [8], density perturbations on large scales [9], curvature perturbations [10] and the Sachs-Wolfe effect [11], vector perturbations [12], tensor perturbations [13] and CMB anisotropies [14]. In all the cases, the effect has been found to be slightly enhanced from the standard analysis. However, we shall show in the subsequent discussions that the Weyl fluid can have a crucial role in late time cosmologies as well when the bulk is not necessarily empty.

### FRW BRANE, VAIDYA-ADS BULK

Till now we have discussed brane cosmology with empty bulk. However, as already mentioned, the bulk may, in principle contain non-standard model fields, for which the above situation is further generalised. In this case, the bulk metric, for which the FRW geometry on the brane is recovered, is given by a radiative 5D Vaidya anti-de-Sitter black hole (VAdS<sub>5</sub>). In terms of transformed (null) coordinate  $v = t + \int dr/f$ , the bulk metric can be written as

$$dS_5^2 = -f(r, v) dv^2 + 2dr dv + r^2 d\Sigma_3^2 \quad (19)$$

where  $\Sigma_3$  is the 3-space. For a spatially flat brane, the function  $f(r, v)$  is given by

$$f(r, v) = \frac{r^2}{l^2} - \frac{m(v)}{r^2} \quad (20)$$

with the length scale  $l$  related to the bulk (negative) cosmological constant by  $\Lambda_5 = -6/l^2$  and  $m(v)$  is the variable mass of the Vaidya black hole.

In the case of such a radiative bulk, the effective Einstein equation on the brane is further modified to [15, 16]

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa_4^2 T_{\mu\nu} + \kappa_5^4 \mathcal{S}_{\mu\nu} - \mathcal{E}_{\mu\nu} + \mathcal{F}_{\mu\nu} \quad (21)$$

Clearly, in this scenario, the extra terms come from three sectors :

- $\mathcal{S}_{\mu\nu}$  : a quadratic term from brane energy-momentum tensor
- $\mathcal{E}_{\mu\nu}$  : a geometric term from the bulk Weyl tensor projected onto the brane
- $\mathcal{F}_{\mu\nu}$  : a term involving the brane-projection of the bulk energy-density.

The combined effect of the last two terms is related to the sumtotal of the mass of bulk black hole and the radiation field, and this is now the *Weyl fluid* that supplies an additional perfect fluid-like effect to the usual brane perfect fluid.

Here the exclusive contribution from the bulk matter is given by the energy-momentum tensor of a null dust

$$T_{AB}^{\text{bulk}} = \psi q_A q_B \quad (22)$$

where  $q_A$  are now the ingoing null vectors and  $\psi \propto dm/dv$  is the rate of ingoing radial flow to the bulk black hole,  $m(v)$  being the resultant of the masses of VAdS<sub>5</sub> black hole and the radiation field. This type of bulk can exchange energy with the brane along the radial direction. There are extensive study in the literature on the energy-exchange between bulk and brane. (see, for example, [15, 16]). Consequently, the brane matter conservation equation is modified to

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = -2\psi \quad (23)$$

which now contains an effect of bulk matter onto the brane and guarantees the null flow from the radiative bulk black hole. Also, the Bianchi identity on the brane  $\nabla^\mu G_{\mu\nu} = 0$  leads to the equation governing the evolution of the Weyl fluid

$$\dot{\rho}^* + 4\frac{\dot{a}}{a}\rho^* = 2\psi - \frac{2\kappa_5^2}{3\kappa_4^2} \left[ \dot{\psi} + 3\frac{\dot{a}}{a}\psi \right] \quad (24)$$

The difference of the above equation with Eq (15) for empty bulk is remarkable. For a bulk with matter, the coupling term involving  $\psi$  determines the nature of the Weyl fluid. A general expression for this is now given by [16]

$$\rho^* = \frac{C(\tau)}{a^4} \quad (25)$$

$\tau$  being the proper time on the brane. Clearly, when the bulk is not strictly matter-free, then the Weyl parameter  $C(\tau)$  is no longer a constant, and consequently,  $\rho^*$  no longer behaves like radiation. It is only when the bulk is empty, we get back the radiation-like behaviour of the Weyl fluid from Eq (24).

It is obvious from the preceding discussion that the Vaidya-AdS bulk scenario is so far the most generalised description of the braneworlds for which cosmology of the 4D world is relevant. Thus, this is the most general *generalised RS II braneworld* scenario.

The components of the effective energy-momentum tensor  $T_{\mu\nu}^{\text{eff}}$  are now given by [17]

$$\rho^{\text{eff}} = \rho + \frac{\rho^2}{2\lambda_b} + \frac{C(\tau)}{a^4} \quad (26)$$

$$p^{\text{eff}} = p + \frac{\rho}{2\lambda_b}(\rho + 2p) + \frac{C(\tau)}{3a^4} \quad (27)$$

As before, for a Vaidya-AdS<sub>5</sub> bulk compatible to FRW geometry on the brane, both  $q_\mu^{\text{eff}}$  and  $\pi_{\mu\nu}^{\text{eff}}$  vanish. We are thus left with a perfect fluid like effect on the brane,

constituted of brane and bulk matter-energy and bulk geometry, with an evolving Weyl fluid.

The final modifications to the Friedmann equation and the covariant Raychaudhuri equation on the brane are due to the collective effect of the embedding geometry, brane matter and bulk matter. In terms of the brane and bulk quantities, these generalised equations read [18]

$$H^2 = \frac{\kappa_4^2}{3}\rho^{\text{eff}} + \frac{\Lambda}{3} - \frac{k}{a^2} \quad (28)$$

$$\dot{H} = -\frac{\kappa_4^2}{2}(\rho^{\text{eff}} + p^{\text{eff}}) + \frac{k}{a^2} - \frac{\kappa_5^2}{3}\psi \quad (29)$$

It is worth mentioning that these equations are radically different from those for empty bulk due to the presence of an evolving Weyl parameter  $C(\tau)$ . So, they should, in principle, give rise to new physics on the brane.

### “NEWTONIAN” PERTURBATIONS

Given the modified Friedmann equations, we will now engage ourselves in studying cosmological perturbations on the brane. Following Newtonian analysis of perturbations from gravitational instability, we will demonstrate that the growth of perturbations of the Weyl fluid can take care of the fluctuations required by the inhomogeneous local universe, and thus, have the potentiality to explain structure formation without the need for dark matter [17].

The equations of hydrodynamics now involve the quadratic brane correction and the Weyl fluid correction to the brane perfect fluid. In terms of the effective quantities, these equations are given by

$$\frac{\partial \rho^{\text{eff}}}{\partial t} + \vec{\nabla} \cdot (\rho^{\text{eff}} \vec{v}^{\text{eff}}) = 0 \quad (30)$$

$$\frac{\partial \vec{v}^{\text{eff}}}{\partial t} + (\vec{v}^{\text{eff}} \cdot \vec{\nabla}) \vec{v}^{\text{eff}} = -\frac{\vec{\nabla} p^{\text{eff}}}{\rho^{\text{eff}}} - \vec{\nabla} \Phi^{\text{eff}} \quad (31)$$

$$\nabla^2 \Phi^{\text{eff}} = 4\pi G \rho^{\text{eff}} \quad (32)$$

where  $\vec{v}^{\text{eff}}$  is the velocity field in the *effective* perfect fluid. It should be noted that the term  $\Phi^{\text{eff}}$  is not the usual Newtonian potential but the effective gravitational potential which is the resultant effect of the brane and bulk parameters in the form of effective quantities.

Let us now consider small perturbation about the initial unperturbed effective quantities. Perturbation on the effective density (the so-called *effective density contrast*) is given by

$$\rho^{\text{eff}}(\vec{x}, \tau) = \bar{\rho}^{\text{eff}}(\tau)(1 + \delta^{\text{eff}}(\vec{x}, \tau)) \quad (33)$$

whereas the perturbation in the effective gravitational potential is

$$\Phi^{\text{eff}}(\vec{x}, \tau) = \Phi_0^{\text{eff}} + \phi^{\text{eff}} \quad (34)$$

where  $\bar{\rho}^{\text{eff}}(\tau)$  and  $\Phi_0^{\text{eff}}$  are respectively the unperturbed effective density and effective potential and  $\delta^{\text{eff}}$  and  $\phi^{\text{eff}}$  are their corresponding fluctuations. For completion, we mention here that a perturbation in the Weyl fluid would mean a perturbation on the bulk geometry. There is a possibility that this may destabilise the brane itself. In this context, however, we assume that the brane remains stable even after such perturbations, which can only be guaranteed if one analyses the effects of perturbations on the full 5D bulk metric.

Expressing in terms of comoving coordinates  $v^{\text{eff}} = \dot{a} r + u^{\text{eff}}$  and neglecting terms of second or higher order in the equations we arrive at the simplified perturbation equations

$$\frac{\partial \delta^{\text{eff}}}{\partial \tau} + \frac{1}{a} \vec{\nabla}_r \cdot \vec{u}^{\text{eff}} = 0 \quad (35)$$

$$\frac{\partial \vec{u}^{\text{eff}}}{\partial \tau} + \frac{\dot{a}}{a} \vec{u}^{\text{eff}} = -\frac{1}{a} \frac{\vec{\nabla}_r p^{\text{eff}}}{\bar{\rho}^{\text{eff}}} - \frac{1}{a} \vec{\nabla}_r \phi^{\text{eff}} \quad (36)$$

$$\nabla_r^2 \phi^{\text{eff}} = 4\pi G a^2 \bar{\rho}^{\text{eff}} \delta^{\text{eff}} \quad (37)$$

The above set of equations have a unique solution given by

$$\delta^{\text{eff}}(\vec{x}, \tau) = \sum \delta_k^{\text{eff}}(\tau) e^{i \vec{k} \cdot \vec{x}} \quad (38)$$

From now on, we shall express the perturbations in terms of the inverse Fourier transform of the above:

$$\delta_k^{\text{eff}}(\tau) = \frac{1}{V} \int \delta^{\text{eff}}(\vec{x}, \tau) e^{-i \vec{k} \cdot \vec{x}} d^3 \vec{x} \quad (39)$$

For a barotropic fluid, the effective pressure is a function of the effective density only. Hence, equations (35) - (37), with the Fourier mode solution, transform into a linear second order differential equation for the perturbation  $\delta_k^{\text{eff}}$

$$\frac{d^2 \delta_k^{\text{eff}}}{d\tau^2} + 2 \frac{\dot{a}}{a} \frac{d \delta_k^{\text{eff}}}{d\tau} - \left[ 4\pi G \bar{\rho}^{\text{eff}} - \left( \frac{c_{\text{eff}}^2 k}{a} \right)^2 \right] \delta_k^{\text{eff}} = 0 \quad (40)$$

where  $c_{\text{eff}}^2$  is the square of the effective sound speed which is given by

$$c_{\text{eff}}^2 = \frac{\dot{p}^{\text{eff}}}{\dot{\rho}^{\text{eff}}} = \left[ c_s^2 + \frac{\rho + p}{\rho + \lambda_b} + \frac{4\rho^*}{9(\rho + p)(1 + \rho/\lambda_b)} \right] \times \left[ 1 + \frac{4\rho^*}{3(\rho + p)(1 + \rho/\lambda_b)} \right]^{-1} \quad (41)$$

The main job now is to analyse the second order differential equation (40) for the effective perturbation. The perturbations of the effective fluid will grow at late times and will account for the required amount of gravitational instability only if the Weyl density redshifts more slowly than baryonic matter density, so that it can dominate over matter at late times. Recall that for a Sch-AdS

bulk (empty bulk), the Weyl fluid is strictly radiation-like, hence it redshifts at a faster rate than baryonic matter, eventually being negligible at late times. But for the VAdS bulk (radiative bulk), the nature and evolution of the Weyl fluid is governed by Eq (24). This equation can be written conveniently as

$$\dot{\rho}^* + 4\frac{\dot{a}}{a}\rho^* = \mathcal{Q} \quad (42)$$

where  $\mathcal{Q}$  is a coupling term which is determined by the projected bulk energy density  $\psi$ . Since the bulk informations are not known *ab initio*, this coupling term is given by a physically reasonable and consistent ansatz. Certain earlier attempt of choosing an ansatz are listed in [18, 21, 22]. We note here that, in order to study the evolution of the Weyl fluid as function of the scale factor, the general ansatz for the coupling term should be of the form [17]

$$\mathcal{Q} = \alpha H \rho^* \quad (43)$$

with  $\alpha > 0$ . For this type of ansatz, the Weyl fluid behaves as

$$\rho^* \propto \frac{1}{a^{(4-\alpha)}} \quad (44)$$

Consequently, Eq (25) reveals that the Weyl parameter is given by

$$C(\tau) = C_0 a^\alpha(\tau) \quad (45)$$

where  $C_0$  is its initial value at the matter-dominated epoch. Note that the Weyl fluid is strictly radiation-like only if  $\alpha = 0$ , for which we recover the Sch-AdS bulk scenario, which is a special case of this generalised description. So, in general, the nature of the Weyl fluid depends on the coupling strength  $\alpha$ . In order that the Weyl fluid dominates over matter the coupling strength  $\alpha$  should have a value within the range  $1 < \alpha < 4$ , as obtained from this ‘‘Newtonian’’ analysis. Later, we shall put more stringent bounds on the value of this parameter from relativistic analysis as well as from observational ground.

Let us now get back to the the perturbation equation (40) of the effective density which is a sum-total of three quantities. As discussed before, the quadratic term being negligible at late times, the effective density at late times is practically given by

$$\rho^{\text{eff}} \approx \rho^{(b)} + \rho^* \quad (46)$$

where  $\rho^{(b)}$  is the baryonic density. Thus, along with the usual matter density, here we have an additional (Weyl) density contributing to the total density that governs the perturbation equation. This Weyl density, being geometric, is essentially non-baryonic. Consequently, we decompose Eq (40) by separating the baryonic (matter) part

from the non-baryonic (Weyl) part, which results in two separate equations, one each for each of the fluids

$$\frac{d^2\delta^{(b)}}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{d\delta^{(b)}}{d\tau} = 4\pi G\bar{\rho}^{(b)}\delta^{(b)} + 4\pi G\bar{\rho}^*\delta^* \quad (47)$$

$$\frac{d^2\delta^*}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{d\delta^*}{d\tau} = 4\pi G\bar{\rho}^*\delta^* + 4\pi G\bar{\rho}^{(b)}\delta^{(b)} \quad (48)$$

where  $\delta^{(b)}$  and  $\delta^*$  are the fluctuations of baryonic matter and Weyl fluid respectively. In the above equations, the term involving sound speed has been neglected as we are interested only in the growing mode fluctuations. With  $\Omega^{(b)} \ll \Omega^*$ , the relevant growing mode solution for Eq (48) can be expressed as a function of the redshift as

$$\delta^*(z) = \delta^*(0)(1+z)^{-1} \quad (49)$$

which, when put back into the fluctuation equation (47) of baryonic density, gives

$$\frac{d^2\delta^{(b)}}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{d\delta^{(b)}}{d\tau} = 4\pi G\bar{\rho}^*\delta^*(0)(1+z)^{-1} \quad (50)$$

Since the late time behaviour of the expansion of the universe in RS II is the same as the standard cosmological solution for the scale factor [15, 20], for a spatially flat ( $k = 0$ ) brane, we have the scale factor at late time

$$a(\tau) = \left(\frac{3}{2}H_0\tau\right)^{2/3(w+1)} \quad (51)$$

Considering  $\Omega^{\text{tot}} \approx \Omega^* \approx 1$  at present time, Eq (50) simplifies to

$$a^{3/2}\frac{d}{da}\left(a^{-1/2}\frac{d\delta^{(b)}}{da}\right) + 2\frac{d\delta^{(b)}}{da} = \frac{3}{2}\delta^*(0) \quad (52)$$

which readily gives a solution of the form [17]

$$\delta^{(b)}(z) = \delta^*(z)\left(1 - \frac{1+z}{1+z_N}\right) \quad (53)$$

where the scale factor is related to the redshift function by  $a \propto (1+z)^{-1}$ .

Eq (53) reveals that at a redshift close to  $z_N$ , the baryonic fluctuation  $\delta^{(b)}$  almost vanishes but the Weyl fluctuation  $\delta^*$  still remains finite. This implies that even if the baryonic fluctuation is very small at a redshift of  $z_N \approx 1000$ , as confirmed by CMB data [19], the fluctuations of the Weyl fluid had a finite amplitude during that time. On the other hand, at a redshift much less than  $z_N$  the baryonic matter fluctuations are of equal amplitude as the Weyl fluid fluctuations. This is precisely what is required to explain the formation of structures we see today.

The bottomline here is that in this analysis, we get a result which resembles standard cosmological relation, but has an essential distinction that nowhere we have to

introduce dark matter by hand. Rather, here the fluctuations are governed by the Weyl density fluctuation  $\delta^*$  which is a product of the embedding geometry via a modified Einstein equation in the braneworld scenario.

Another point is worth mentioning here. The *effective* equation of state parameter

$$w^{\text{eff}} = \frac{p^{\text{eff}}}{\rho^{\text{eff}}} = \frac{p^{(b)} + \rho^{(b)}(\rho^{(b)} + 2p^{(b)})/2\lambda_b + C(\tau)/3a^4}{\rho^{(b)} + \rho^{(b)2}/2\lambda_b + C(\tau)/a^4} \quad (54)$$

reduces to (in this context of matter-dominated era)

$$w^{\text{eff}} \approx \frac{1}{3(1 + Ca^{1-\alpha})} \quad (55)$$

which bears significant difference from the equation of state of cold dark matter ( $w = 0$ ). Thus, the theory gives rise to a geometric fluid which is very different from dark matter in origin and nature but has the potentiality to play the role of dark matter in cosmological context.

## RELATIVISTIC PERTURBATIONS

Let us now proceed further to study relativistic perturbations on the brane based on the above analysis. The scenario, in brief, is as follows: Here the cosmological dynamics is governed by a two-fluid system, out of which one is the material fluid  $\rho^{(b)}$ , which is the ordinary (baryonic) matter, and another a geometric fluid  $\rho^*$ , termed as the Weyl fluid. These two fluids interact and exchange energy between them in such a way that the total (effective) density on the brane is given by  $\rho = \rho^{(b)} + \rho^*$ . [23]

Given this scenario, the comoving fractional gradients of density and expansion are expressed, following usual general relativity, as

$$\Delta_\mu^{(i)} = \frac{a}{\rho^{(i)}} D_\mu \rho^{(i)} \quad (56)$$

$$Z_\mu = a D_\mu \Theta \quad (57)$$

$$\Delta_\mu = \frac{a}{\rho} D_\mu \rho \quad (58)$$

Further, the conservation equations (23) and (24) can be written composedly as

$$\dot{\rho}^{(i)} + \Theta(\rho^{(i)} + p^{(i)}) = I^{(i)} \quad (59)$$

where a superscript  $(i)$  denotes the quantities for the  $i$ -th fluid and  $I^{(i)}$  is the corresponding interaction term. Written explicitly in the bulk-brane scenario discussed here, the interaction terms are:

$$I^{(b)} = -2\psi \quad (60)$$

$$I^* = 2\psi - \frac{2}{3} \left( \frac{\kappa_5}{\kappa} \right)^2 \left( \dot{\psi} + 3 \frac{\dot{a}}{a} \psi \right) \quad (61)$$

In what follows we shall restrict ourselves to the discussion of the Einstein-de Sitter brane universe for which  $\Omega^* = 1, \Omega_\Lambda = 0$ . Here, with the above notations, the linearised evolution equations turn out to be

$$\begin{aligned} \dot{\Delta}_\mu^{(i)} &= \left( 3Hw^{(i)} - \frac{I^{(i)}}{\rho^{(i)}} \right) \Delta_\mu^{(i)} - (1 + w^{(i)}) Z_\mu \\ &\quad - \frac{c_s^2 I^{(i)}}{\rho^{(i)}(1+w)} \Delta_\mu - \frac{3aH I_\mu^{(i)}}{\rho^{(i)}} + \frac{a}{\rho^{(i)}} D_\mu I^{(i)} \quad (62) \\ \dot{Z}_\mu + 2HZ_\mu &= -\frac{\kappa^2}{2} \rho \Delta - \frac{c_s^2}{1+w} D_\mu D^\nu \Delta_\nu \\ &\quad + \frac{\kappa_5^2 \psi}{1+w} c_s^2 \Delta_\mu - a \kappa_5^2 D_\mu \psi \quad (63) \end{aligned}$$

We shall now try to express the above equations in terms of covariant quantities. These density perturbations are governed by the fluctuation of the following covariant projections

$$\Delta^{(i)} = a D^\mu \Delta_\mu^{(i)}, \Delta = a D^\mu \Delta_\mu, Z = a D^\mu Z_\mu \quad (64)$$

The covariant density perturbation equations on the brane, when expressed in terms of the above covariant quantities, turn out to be

$$\begin{aligned} \dot{\Delta}^{(i)} &= \left( 3Hw^{(i)} - \frac{I^{(i)}}{\rho^{(i)}} \right) \Delta^{(i)} - (1 + w^{(i)}) Z \\ &\quad - \frac{c_s^2 I^{(i)}}{\rho^{(i)}(1+w)} \Delta - \frac{3a^2 H D^\mu I_\mu^{(i)}}{\rho^{(i)}} + \frac{a^2}{\rho^{(i)}} D^2 I^{(i)} \quad (65) \\ \dot{Z} + 2HZ &= -\frac{\kappa^2}{2} \rho \Delta - \frac{a c_s^2}{1+w} D^2 \Delta \\ &\quad + \frac{\kappa_5^2 \psi}{1+w} c_s^2 \Delta - a^2 \kappa_5^2 D^2 \psi \quad (66) \end{aligned}$$

Obviously, the equations are too complicated and it is almost impossible to have a possible solution from these complicated equations *a priori*. However, the equations turn out to be tractable if we incorporate certain simplifications following physical arguments. We have seen in the case of Newtonian analysis that for a radiative bulk the Weyl fluid, in general, evolves as

$$\rho^* = C_0 a^{-(4-\alpha)} \quad (67)$$

with the parameter  $\alpha$  in the range  $1 < \alpha < 4$ . We now consider  $\psi$  to be a function of time only. This essentially means that we are considering only the time-evolution of the bulk-brane scenario, which is relevant for perturbation analysis. We further assume that the energy exchange between the two fluids is in equilibrium. This basically describes the late time behaviour. Once again this is consistent so far as density perturbations are concerned. Consequently, the Weyl fluid now behaves as

$$\rho^* \propto a^{-3/2} \quad (68)$$

with the parameter  $\alpha = \frac{5}{2}$ . This readily suggests that the Weyl fluid redshifts more slowly than ordinary matter and hence, can dominate over matter at late times, reflecting one of the fundamental properties of dark matter. This also provides a more stringent bound for the value of  $\alpha$  from theoretical ground alone. Later, we shall confront this value with observations.

With the above results, the evolution equation for the Weyl fluid at late times is radically simplified by using  $\Delta^{(b)} \ll \Delta^*$ , which can be recast in convenient form as

$$\ddot{\Delta}^* + \frac{A}{t}\dot{\Delta}^* - \left(\frac{B}{t} + \frac{C}{t^2}\right)\Delta^* = 0 \quad (69)$$

where the constants  $A, B, C$  are readily determinable.

The above equation for  $\Delta^*$  turns out to be somewhat tractable. One of its solutions is given by [23]

$$\Delta^* \sim t^{\frac{1}{2} - \frac{A}{2}} \text{Bessel}I \left[ \sqrt{1 - 2A + A^2 + 4C}, 2\sqrt{B}\sqrt{t} \right] \quad (70)$$

We know that the Bessel function is a growing function. Therefore, the evolution equation for the Weyl fluid, indeed, shows a growing mode solution, which is required to explain the growth of perturbations at late times.

Thus, even the relativistic perturbations show that so far as theoretical results are concerned, the Weyl fluid can explain structure formation.

## CONFRONTATION WITH OBSERVATIONS

An important issue is to confront this theoretical model with observations. Recently there has been some study in this direction [24], although an extensive study still remains as an open issue. In this section, we shall very briefly mention what has been done in this direction so far.

In terms of dimensionless parameters

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \Omega_\rho = \frac{\kappa^2 \rho_0}{3H_0^2}, \Omega_* = \frac{2C_0}{a_0^{4-\alpha} H_0^2}, \Omega_\lambda = \frac{\kappa^2 \rho_0^2}{6\lambda H_0^2} \quad (71)$$

with the total density satisfying the critical value

$$\Omega_{\text{tot}} = \sum_i \Omega_i = 1 \quad (72)$$

the Friedmann equation (including the contribution from brane tension) can be re-written as

$$\frac{H^2}{H_0^2} = \Omega_\Lambda + \Omega_\rho \frac{a_0^3}{a^3} + \Omega_* \frac{a_0^{4-\alpha}}{a^{4-\alpha}} + \Omega_\lambda \frac{a_0^6}{a^6} \quad (73)$$

The luminosity distance for FRW branes is now given by

$$d_L(z) = \frac{(1+z)a_0}{H_0} \int_a^{a_0} \frac{ada}{[\Omega_\Lambda a^6 + \Omega_\rho a_0^3 a^3 + \Omega_* a_0^{4-\alpha} a^{\alpha+2} + \Omega_\lambda a_0^6]^{1/2}}$$

For  $\Omega_\lambda \rightarrow 0$  relevant for late-time cosmology, this integral can be evaluated as

$$d_L^{\Lambda\lambda*} = d_L^{\Lambda\text{CDM}} + \Omega_* I_* \quad (74)$$

where  $I_*$  is a function having elliptic integrals of 1st and 2nd kind, which have exact analytical expressions.

Comparing this with standard  $\Lambda$ CDM scenario, one finds that a certain amount of Weyl fluid with  $2 \leq \alpha \leq 3$  is in nice agreement with SNe data [24].

It is important to mention here that in the relativistic analysis we have found that  $\alpha = \frac{5}{2}$ , which falls in this region, and hence, the braneworld model of perturbations is so far an observationally accepted model. However, as already mentioned, an extensive study in this direction remains as a very crucial open issue, which, we hope, will be addressed in details in near future.

## SUMMARY AND OUTLOOK

We have discussed certain features of cosmology in a generalised RS II braneworld scenario, where the bulk is either a Schwarzschild-anti de Sitter or a radiative Vaidya-anti de Sitter black hole. We have shown that the theory leads to a modified version of the brane Friedmann equations. Specifically, the local corrections to the Friedmann equations are manifest via a quadratic contribution from the brane perfect fluid whereas the nonlocal corrections supply a Weyl fluid which arises as an effect of the bulk-brane geometry. We have investigated for the role of each of the terms for early times as well as for late time cosmologies. Further, we have shown that the Weyl fluid plays a crucial role in late time cosmology, for the most general bulk metric. We have demonstrated that its presence radically changes the perturbation equations, and have discussed some of the implications of fluctuations involving it. The fluctuations are found to grow at late times and thus, may take care of the large amount of inhomogeneities observed in the local universe. We also mentioned some observable sides of this model.

Some future directions related to braneworld cosmology involve both theoretical and observational aspects. In the theoretical side, a thorough study of different parameters related to cosmological perturbation need to be performed. In the observational side, confronting this model with observations in more details will test the theory in a more conclusive way. Last, but not the least, the expansion history of the universe from these modified equations can be progressed further, which may lead to more interesting results and may even be investigated to find out any possible link with observations.

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